Laplace Transforms

Μάθημα 6ο (Αναπλήρωση 13-4-2020)

Επανάληψη

Laplace transforms

The Laplace transform of an expression $f(t)$ is denoted by $L\{f(t)\}$ and is defined as the semi-infinite integral

$$L\{f(t)\} = \int_{t=0}^{\infty} f(t)e^{-st}\,dt \qquad (1)$$

$$L\{f(t)\} = \int_{t=0}^{\infty} f(t)e^{-st}\,dt = F(s)$$

Τυπολόγιο

$$L\{a\} = \frac{a}{s}; \qquad L\{e^{at}\} = \frac{1}{s-a}; \qquad L\{t^n\} = \frac{n!}{s^{n+1}}$$

$$L\{\sin at\} = \frac{a}{s^2+a^2}; \qquad L\{\cos at\} = \frac{s}{s^2+a^2}$$

$$L\{\sinh at\} = \frac{a}{s^2-a^2}; \qquad L\{\cosh at\} = \frac{s}{s^2-a^2}$$

Θεωρήματα

(2) *The transform of an expression that is multiplied by a constant is the constant multiplied by the transform of the expression. That is*

$$L\{kf(t)\} = kL\{f(t)\}$$

(1) *The transform of a sum (or difference) of expressions is the sum (or difference) of the individual transforms. That is*

$$L\{f(t) \pm g(t)\} = L\{f(t)\} \pm L\{g(t)\}$$

Theorem 1 The first shift theorem

The first shift theorem states that if $L\{f(t)\} = F(s)$ then $L\{e^{-at}f(t)\} = F(s+a)$

Because $L\{e^{-at}f(t)\} = \int_{t=0}^{\infty} e^{-at}f(t)e^{-st}\,\mathrm{d}t = \int_{t=0}^{\infty} f(t)e^{-(s+a)t}\,\mathrm{d}t = F(s+a)$

Theorem 2 Multiplying by t and t^n

If $L\{f(t)\} = F(s)$ then $L\{tf(t)\} = -F'(s)$

Because
$$L\{tf(t)\} = \int_{t=0}^{\infty} tf(t)e^{-st}\,\mathrm{d}t = \int_{t=0}^{\infty} f(t)\left(-\frac{\mathrm{d}e^{-st}}{\mathrm{d}s}\right)\mathrm{d}t$$
$$= -\frac{\mathrm{d}}{\mathrm{d}s}\int_{t=0}^{\infty} f(t)e^{-st}\,\mathrm{d}t = -F'(s)$$

So, in general, if $L\{f(t)\} = F(s)$, then

$$L\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}\{F(s)\}$$

Theorem 3 Dividing by t

If $L\{f(t)\} = F(s)$ then $L\left\{\frac{f(t)}{t}\right\} = \int_{\sigma=s}^{\infty} F(\sigma)\, d\sigma$

provided $\lim_{t \to 0} \left(\frac{f(t)}{t}\right)$ exists.

1 *Standard transforms*

$f(t)$	$L\{f(t)\} = F(s)$
a	$\frac{a}{s}$
e^{at}	$\frac{1}{s-a}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
t^n	$\frac{n!}{s^{n+1}}$

(n a positive integer)

2 *Theorem 1* The first shift theorem

If $L\{f(t)\} = F(s)$, then $L\{e^{-at}f(t)\} = F(s+a)$

3 *Theorem 2* Multiplying by t

If $L\{f(t)\} = F(s)$, then $L\{tf(t)\} = -\frac{\mathrm{d}}{\mathrm{d}s}\{F(s)\}$

4 *Theorem 3* Dividing by t

If $L\{f(t)\} = F(s)$, then $L\left\{\frac{f(t)}{t}\right\} = \int_{\sigma=s}^{\infty} F(\sigma)\,\mathrm{d}\sigma$

provided $\lim_{t\to 0}\left\{\frac{f(t)}{t}\right\}$ exists.

Inverse transforms

Here we have the reverse process, i.e. given a Laplace transform, we have to find the function of t to which it belongs.

$$\frac{a}{s^2 + a^2}$$

$$L^{-1}\left\{\frac{a}{s^2 + a^2}\right\}$$

$$\sin at$$

$$L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$e^{2t}$$

$$L^{-1}\left\{\frac{s}{s^2+25}\right\}$$

$$\cos 5t$$

$$L^{-1}\left\{\frac{4}{s}\right\}$$

$$4$$

$$L^{-1}\left\{\frac{12}{s^2-9}\right\}$$

$$4\sinh 3t$$

$$L^{-1}\left\{\frac{3s+1}{s^2-s-6}\right\}$$

$$\frac{1}{s+2}+\frac{2}{s-3}$$

$$L^{-1}\left\{\frac{3s+1}{s^2-s-6}\right\}=L^{-1}\left\{\frac{1}{s+2}+\frac{2}{s-3}\right\}$$

$$e^{-2t}+2e^{3t}$$

partial fractions

Rules of partial fractions

1 The numerator must be of lower degree than the denominator. This is usually the case in Laplace transforms. If it is not, then we first divide out.

2 Factorise the denominator into its prime factors. These determine the shapes of the partial fractions.

3 A linear factor $(s+a)$ gives a partial fraction $\frac{A}{s+a}$ where A is a constant to be determined.

4 A repeated factor $(s+a)^2$ gives $\frac{A}{(s+a)}+\frac{B}{(s+a)^2}$.

5 Similarly $(s+a)^3$ gives $\frac{A}{(s+a)}+\frac{B}{(s+a)^2}+\frac{C}{(s+a)^3}$.

6 A quadratic factor (s^2+ps+q) gives $\frac{Ps+Q}{s^2+ps+q}$.

7 Repeated quadratic factors $(s^2+ps+q)^2$ give

$$\frac{Ps+Q}{s^2+ps+q}+\frac{Rs+T}{(s^2+ps+q)^2}.$$

$$\frac{s-19}{(s+2)(s-5)}$$

$$\frac{A}{s+2}+\frac{B}{s-5}$$

$$\frac{3s^2-4s+11}{(s+3)(s-2)^2}$$

$$\frac{A}{s+3}+\frac{B}{(s-2)}+\frac{C}{(s-2)^2}$$

Example 1

$$L^{-1}\left\{\frac{5s+1}{s^2-s-12}\right\}$$

(a) First we check that the numerator is of lower degree than the denominator. In fact, this is so.

(b) Factorise the denominator $\frac{5s+1}{s^2-s-12}=\frac{5s+1}{(s-4)(s+3)}$

(c) Then the partial fractions are of the form

$$\frac{A}{s-4}+\frac{B}{s+3}$$

$$\frac{5s+1}{s^2-s-12}\equiv\frac{A}{s-4}+\frac{B}{s+3}$$

$$s^2 - s - 12$$

$$(s-4)(s+3)$$

$$5s + 1 \equiv A(s+3) + B(s-4)$$

This is also an identity and true for any value of s

Let $(s-4) = 0$, i.e. $s = 4$

$$21 = A(7) + B(0)$$

$$A = 3$$

$$(s + 3) = 0, \text{ i.e. } s = -3$$

$$B = 2$$

$$\frac{5s + 1}{s^2 - s - 12} \equiv \frac{3}{s - 4} + \frac{2}{s + 3}$$

$$L^{-1}\left\{\frac{5s + 1}{s^2 - s - 12}\right\}$$

$$3e^{4t} + 2e^{-3t}$$

Example 2

$$L^{-1}\left\{\frac{9s-8}{s^2-2s}\right\}$$

- Numerator of first degree; denonominator of second degree. Therefore rule satisfied.
- $$\frac{9s-8}{s(s-2)} \equiv \frac{A}{s} + \frac{B}{s-2}$$
- Multiply by $s(s-2)$
- $$9s-8 = A(s-2) + Bs.$$
- Put $s = 0$

$$-8 = A(-2) + B(0)$$

$$A = 4.$$

$s - 2 = 0$, i.e. $s = 2$.

$$10 = A(0) + B(2)$$

$$B = 5$$

$$f(t) = L^{-1}\left\{\frac{4}{s} + \frac{5}{s-2}\right\} = 4 + 5e^{2t}$$

Example 3

$$F(s) = \frac{s^2 - 15s + 41}{(s+2)(s-3)^2}$$

$$\frac{A}{s+2} + \frac{B}{s-3} + \frac{C}{(s-3)^2}$$

we multiply throughout by $(s+2)(s-3)^2$

$$s^2 - 15s + 41 \equiv A(s-3)^2 + B(s+2)(s-3) + C(s+2)$$

$$(s-3) = 0$$

$$(s+2) = 0$$

$$A = 3 \text{ and } C = 1$$

$$1 = A + B \qquad B = -2$$

$$\frac{s^2 - 15s + 41}{(s+2)(s-3)^2} = \frac{3}{s+2} - \frac{2}{s-3} + \frac{1}{(s-3)^2}$$

$$L^{-1}\left\{\frac{3}{s+2}\right\}$$

$$L^{-1}\left\{\frac{2}{s-3}\right\}$$

$$3e^{-2t}$$

$$2e^{3t}$$

$$L^{-1}\left\{\frac{1}{(s-3)^2}\right\}$$

$$L^{-1}\left\{\frac{1}{s^2}\right\}$$

by Theorem 1

if $L\{f(t)\} = F(s)$ then $L\{e^{-at}f(t)\} = F(s+a)$

$\frac{1}{(s-3)^2}$ is like $\frac{1}{s^2}$ with s replaced by $(s-3)$ i.e. $a = -3$.

$$L^{-1}\left\{\frac{1}{(s-3)^2}\right\} = te^{3t}$$

$$L^{-1}\left\{\frac{s^2-15s+41}{(s+2)(s-3)^2}\right\} = 3e^{-2t} + 2e^{3t} + te^{3t}$$

Example 4

Determine $L^{-1}\left\{\dfrac{4s^2-5s+6}{(s+1)(s^2+4)}\right\}$.

$$\frac{4s^2-5s+6}{(s+1)(s^2+4)} \equiv \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

$$4s^2-5s+6 \equiv A(s^2+4) + (Bs+C)(s+1)$$

$$(s+1)=0, \text{ i.e. } s=-1$$

$$15=5A \quad \therefore\ A=3$$

Equate coefficients of highest power, i.e. s^2

$$4=A+B \quad \therefore\ 4=3+B \quad \therefore\ B=1$$

We now equate the lowest power on each side,

$$6=4A+C \quad \therefore\ 6=12+C \quad \therefore\ C=-6$$

$$f(t)=3e^{-t}+\cos 2t-3\sin 2t$$

$$L\{f(t)\} = \frac{3}{s+1} + \frac{s}{s^2+4} - \frac{6}{s^2+4}$$

$$\therefore \; f(t) = 3e^{-t} + \cos 2t - 3\sin 2t$$

Table of inverse transforms

$F(s)$	$f(t)$	
$\frac{a}{s}$	a	
$\frac{1}{s+a}$	e^{-at}	
$\frac{n!}{s^{n+1}}$	t^n	(n a positive integer)
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$	(n a positive integer)
$\frac{a}{s^2+a^2}$	$\sin at$	
$\frac{s}{s^2+a^2}$	$\cos at$	
$\frac{a}{s^2-a^2}$	$\sinh at$	
$\frac{s}{s^2-a^2}$	$\cosh at$	

Exercise

1 Find the inverse transforms of

(a) $\dfrac{1}{2s-3}$; (b) $\dfrac{5}{(s-4)^3}$; (c) $\dfrac{3s+4}{s^2+9}$.

2 Express in partial fractions

(a) $\dfrac{22s+16}{(s+1)(s-2)(s+3)}$; (b) $\dfrac{s^2-11s+6}{(s+1)(s-2)^2}$.

3 Determine

(a) $L^{-1}\left\{\dfrac{4s^2-17s-24}{s(s+3)(s-4)}\right\}$; (b) $L^{-1}\left\{\dfrac{5s^2-4s-7}{(s-3)(s^2+4)}\right\}$.