Laplace Transforms

Μάθημα 5ο

Επανάληψη

Laplace transforms

The Laplace transform of an expression $f(t)$ is denoted by $L\{f(t)\}$ and is defined as the semi-infinite integral

$$L\{f(t)\} = \int_{t=0}^{\infty} f(t)e^{-st}\,dt \qquad (1)$$

$$L\{f(t)\} = \int_{t=0}^{\infty} f(t)e^{-st}\,dt = F(s)$$

Τυπολόγιο

$$L\{a\} = \frac{a}{s}; \qquad L\{e^{at}\} = \frac{1}{s-a}; \qquad L\{t^n\} = \frac{n!}{s^{n+1}}$$

$$L\{\sin at\} = \frac{a}{s^2+a^2}; \qquad L\{\cos at\} = \frac{s}{s^2+a^2}$$

$$L\{\sinh at\} = \frac{a}{s^2-a^2}; \qquad L\{\cosh at\} = \frac{s}{s^2-a^2}$$

Θεωρήματα

(2) *The transform of an expression that is multiplied by a constant is the constant multiplied by the transform of the expression. That is*

$$L\{kf(t)\} = kL\{f(t)\}$$

(1) *The transform of a sum (or difference) of expressions is the sum (or difference) of the individual transforms. That is*

$$L\{f(t) \pm g(t)\} = L\{f(t)\} \pm L\{g(t)\}$$

Theorem 1 The first shift theorem

The first shift theorem states that if $L\{f(t)\} = F(s)$ then $L\{e^{-at}f(t)\} = F(s+a)$

Because $L\{e^{-at}f(t)\} = \int_{t=0}^{\infty} e^{-at}f(t)e^{-st}\,dt = \int_{t=0}^{\infty} f(t)e^{-(s+a)t}\,dt = F(s+a)$

Παράδειγμα 1

For example $L\{\sin 2t\} = \dfrac{2}{s^2+4}$

then $L\{e^{-3t}\sin 2t\} = \dfrac{2}{(s+3)^2+4} = \dfrac{2}{s^2+6s+13}$

Παράδειγμα 2

Similarly, $L\{t^2\} = \dfrac{2}{s^3}$ $\therefore$ $L\{t^2e^{4t}\} =$

$$\frac{2}{(s-4)^3}$$

Exercise

Determine the following.

1. $L\{e^{-2t}\cosh 3t\}$
2. $L\{2e^{3t}\sin 3t\}$
3. $L\{4te^{-t}\}$
4. $L\{e^{2t}\cos t\}$
5. $L\{e^{3t}\sinh 2t\}$
6. $L\{t^3e^{-4t}\}$

1. $L\{\cosh 3t\} = \dfrac{s}{s^2 - 9}$ $\quad \therefore L\{e^{-2t}\cosh 3t\} = \dfrac{s+2}{(s+2)^2 - 9} = \dfrac{s+2}{s^2 + 4s - 5}$

2. $L\{\sin 3t\} = \dfrac{3}{s^2 + 9}$ $\quad \therefore L\{2e^{3t}\sin 3t\} = \dfrac{6}{(s-3)^2 + 9} = \dfrac{6}{s^2 - 6s + 18}$

3. $L\{4t\} = 4.\dfrac{1}{s^2}$ $\quad \therefore L\{4te^{-t}\} = \dfrac{4}{(s+1)^2}$

4. $L\{\cos t\} = \dfrac{s}{s^2 + 1}$ $\quad \therefore L\{e^{2t}\cos t\} = \dfrac{s-2}{(s-2)^2 + 1} = \dfrac{s-2}{s^2 - 4s + 5}$

5. $L\{\sinh 2t\} = \dfrac{2}{s^2 - 4}$ $\quad \therefore L\{e^{3t}\sinh 2t\} = \dfrac{2}{(s-3)^2 - 4} = \dfrac{2}{s^2 - 6s + 5}$

6. $L\{t^3\} = \dfrac{3!}{s^4}$ $\quad \therefore L\{t^3 e^{-4t}\} = \dfrac{6}{(s+4)^4}$

Theorem 2 Multiplying by t and t^n

If $L\{f(t)\} = F(s)$ then $L\{tf(t)\} = -F'(s)$

$$\text{Because } L\{tf(t)\} = \int_{t=0}^{\infty} tf(t)e^{-st}\,\mathrm{d}t = \int_{t=0}^{\infty} f(t)\left(-\frac{\mathrm{d}e^{-st}}{\mathrm{d}s}\right)\mathrm{d}t$$
$$= -\frac{\mathrm{d}}{\mathrm{d}s}\int_{t=0}^{\infty} f(t)e^{-st}\,\mathrm{d}t = -F'(s)$$

<u>Παράδειγμα 1</u>

For example, $L\{\sin 2t\} = \dfrac{2}{s^2+4}$

$$L\{t\sin 2t\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{2}{s^2+4}\right) = \frac{4s}{(s^2+4)^2}$$

<u>Παράδειγμα 2</u>

$$L\{t\cosh 3t\} =$$

Because $L\{t\cosh 3t\} = -\dfrac{d}{ds}\left(\dfrac{s}{s^2-9}\right) = -\dfrac{(s^2-9)-s(2s)}{(s^2-9)^2} = \dfrac{s^2+9}{(s^2-9)^2}$

We could, if necessary, take this a stage further and find $L\{t^2\cosh 3t\}$

$$L\{t^2\cosh 3t\} = L\{t(t\cosh 3t)\} = -\frac{d}{ds}\left\{\frac{s^2+9}{(s^2-9)^2}\right\}$$
$$= \frac{2s(s^2+27)}{(s^2-9)^3}$$

<u>Παράδειγμα 3</u>

Likewise, starting with $L\{\sin 4t\} = \dfrac{4}{s^2+16}$

$$L\{t\sin 4t\} =$$

$$\frac{8s}{(s^2+16)^2}$$

$$L\{t^2\sin 4t\} =$$

$$\frac{8(3s^2-16)}{(s^2+16)^3}$$

Το Θεώρημα 2 επεκτείνει το εύρος των συναρτήσεων των οποίων μπορούμε να βρούμε τον Μετασχηματισμό Laplace.

So, in general, if $L\{f(t)\} = F(s)$, then

$$L\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}\{F(s)\}$$

Theorem 3 Dividing by *t*

If $L\{f(t)\} = F(s)$ then $L\left\{\frac{f(t)}{t}\right\} = \int_{\sigma=s}^{\infty} F(\sigma)\,\mathrm{d}\sigma$

provided $\underset{t\to 0}{Lim}\left(\frac{f(t)}{t}\right)$ exists.

$$\begin{aligned}
\int_{\sigma=s}^{\infty} F(\sigma)\,\mathrm{d}\sigma &= \int_{\sigma=s}^{\infty}\left\{\int_{t=0}^{\infty} f(t)e^{-\sigma t}\,\mathrm{d}t\right\}\mathrm{d}\sigma \\
&= \int_{t=0}^{\infty}\int_{\sigma=s}^{\infty} f(t)e^{-\sigma t}\,\mathrm{d}\sigma\,\mathrm{d}t \\
&= \int_{t=0}^{\infty} f(t)\left\{\int_{\sigma=s}^{\infty} e^{-\sigma t}\,\mathrm{d}\sigma\right\}\mathrm{d}t \\
&= \int_{t=0}^{\infty} f(t)\frac{e^{-st}}{t}\,\mathrm{d}t \\
&= L\left\{\frac{f(t)}{t}\right\}
\end{aligned}$$

In indeterminate cases, we use L'Hôpital's rule

Example 1

Determine $L\left\{\frac{\sin at}{t}\right\}$

First we test $\lim_{t\to 0}\left\{\frac{\sin at}{t}\right\}=\left\{\frac{0}{0}\right\}=?$

By L'Hôpital's rule, we differentiate top and bottom separately and substitute $t=0$ in the result to ascertain the limit of the new expression.

$$\lim_{t\to 0}\left\{\frac{\sin at}{t}\right\}=\lim_{t\to 0}\left\{\frac{a\cos at}{1}\right\}=a$$

So $L\{\sin at\}=\frac{a}{s^2+a^2}$, therefore

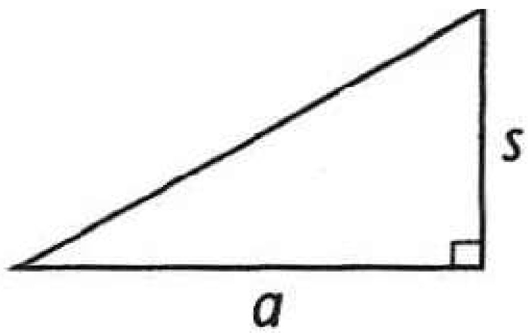
$$\begin{aligned} L\left\{\frac{\sin at}{t}\right\} &= \int_s^\infty \frac{a}{\sigma^2+a^2}\,\mathrm{d}\sigma \\ &= \left[\arctan\left(\frac{\sigma}{a}\right)\right]_s^\infty \\ &= \frac{\pi}{2}-\arctan\left(\frac{s}{a}\right) \\ &= \arctan\left(\frac{a}{s}\right) \end{aligned}$$

So $L\{\sin at\} = \frac{a}{s^2 + a^2}$, therefore $L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{\sigma^2 + a^2}\, d\sigma$

$$= \left[\arctan\left(\frac{\sigma}{a}\right)\right]_s^\infty$$

$$= \frac{\pi}{2} - \arctan\left(\frac{s}{a}\right)$$

$$= \arctan\left(\frac{a}{s}\right)$$



Example 2

Determine $L\left\{\frac{1 - \cos 2t}{t}\right\}$

$$\lim_{t \to 0}\left\{\frac{1 - \cos 2t}{t}\right\} = \frac{1 - 1}{0} = \frac{0}{0} = ?$$

Apply l'Hôpital's rule.

$$\lim_{t\to 0}\left\{\frac{1-\cos 2t}{t}\right\} = \lim_{t\to 0}\left\{\frac{2\sin 2t}{1}\right\} = \frac{0}{1} = 0$$

limit exists.

$$L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$$

Then, by Theorem 3

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \int_{\sigma=s}^{\infty}\left\{\frac{1}{\sigma} - \frac{\sigma}{\sigma^2+4}\right\}d\sigma$$

$$= \left[\ln\sigma - \frac{1}{2}\ln(\sigma^2+4)\right]_{\sigma=s}^{\infty} = \frac{1}{2}\left[\ln\left(\frac{\sigma^2}{\sigma^2+4}\right)\right]_{\sigma=s}^{\infty}$$

When $\sigma \to \infty$, $\ln\left(\frac{\sigma^2}{\sigma^2+4}\right) \to \ln 1 = 0$

Therefore, $L\left\{\frac{1-\cos 2t}{t}\right\} =$

$$L\left\{\frac{1-\cos 2t}{t}\right\} = -\frac{1}{2}\ln\left(\frac{s^2}{s^2+4}\right) = \ln\left(\frac{s^2}{s^2+4}\right)^{-1/2}$$

$$= \ln\sqrt{\frac{s^2+4}{s^2}}$$

1 *Standard transforms*

$f(t)$	$L\{f(t)\} = F(s)$	
a	$\frac{a}{s}$	
e^{at}	$\frac{1}{s-a}$	
$\sin at$	$\frac{a}{s^2+a^2}$	
$\cos at$	$\frac{s}{s^2+a^2}$	
$\sinh at$	$\frac{a}{s^2-a^2}$	
$\cosh at$	$\frac{s}{s^2-a^2}$	
t^n	$\frac{n!}{s^{n+1}}$	(n a positive integer)

2 *Theorem 1* The first shift theorem

If $L\{f(t)\} = F(s)$, then $L\{e^{-at}f(t)\} = F(s+a)$

3 *Theorem 2* Multiplying by t

If $L\{f(t)\} = F(s)$, then $L\{tf(t)\} = -\frac{\mathrm{d}}{\mathrm{d}s}\{F(s)\}$

4 *Theorem 3* Dividing by t

If $L\{f(t)\} = F(s)$, then $L\left\{\frac{f(t)}{t}\right\} = \int_{\sigma=s}^{\infty} F(\sigma)\,\mathrm{d}\sigma$

provided $\lim_{t\to 0}\left\{\frac{f(t)}{t}\right\}$ exists.

Exercise

Determine the Laplace transforms of the following expressions.

1	$\sin 3t$	**6**	$t \cosh 4t$
2	$\cos 2t$	**7**	$t^2 - 3t + 4$
3	e^{4t}	**8**	$\dfrac{e^{3t} - 1}{t}$
4	$6t^2$	**9**	$e^{3t} \cos 4t$
5	$\sinh 3t$	**10**	$t^2 \sin t$

1	$\dfrac{3}{s^2 + 9}$	**6**	$\dfrac{s^2 + 16}{(s^2 - 16)^2}$
2	$\dfrac{s}{s^2 + 4}$	**7**	$\dfrac{1}{s^3}(4s^2 - 3s + 2)$
3	$\dfrac{1}{s - 4}$	**8**	$\ln\left(\dfrac{s}{s - 3}\right)$
4	$\dfrac{12}{s^3}$	**9**	$\dfrac{s - 3}{s^2 - 6s + 25}$
5	$\dfrac{3}{s^2 - 9}$	**10**	$\dfrac{6s^2 - 2}{(s^2 + 1)^3}$